“Baltic Way – 91” mathematical team contest

Tartu, December 14, 1991

1. Find the smallest positive integer n having the property: for any set of n distinct integers $a_1, a_2, \ldots, a_n$ the product of all differences $a_i - a_j$, $i < j$ is divisible by 1991.

2. Prove that there are no positive integers n and $m > 1$ such that $102^{1991} + 103^{1991} = n^m$.

3. There are 20 cats priced from \$12 to \$15 and 20 sacks priced from 10 cents to \$1 for sale (all prices are different). Prove that each of two boys, John and Peter, can buy a cat in a sack paying the same amount of money.

4. Let p be a polynomial with integer coefficients such that $p(-n) < p(n) < n$ for some integer n. Prove that $p(-n) < -n$.

5. For any positive numbers a, b, c prove the inequalities
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geq \frac{9}{a+b+c}.$$

6. Let $[x]$ be the integer part of a number x, and $\{x\} = x - [x]$. Solve the equation
$$[x] \cdot \{x\} = 1991x.$$

7. Let A, B, C be the angles of an acute-angled triangle. Prove the inequality
$$\sin A + \sin B > \cos A + \cos B + \cos C.$$

8. Let a, b, c, d, e be distinct real numbers. Prove that the equation
$$\begin{aligned}&(x-a)(x-b)(x-c)(x-d)\\&+(x-a)(x-b)(x-c)(x-e)\\&+(x-a)(x-b)(x-d)(x-e)\\&+(x-a)(x-c)(x-d)(x-e)\\&+(x-b)(x-c)(x-d)(x-e)=0\end{aligned}$$
has 4 distinct real solutions.

9. Find the number of solutions of the equation $ae^x = x^3$.

10. Express the value of $\sin 3^\circ$ in radicals.

11. All positive integers from 1 to 1 000 000 are divided into two groups consisting of numbers with odd or even sums of digits respectively. Which group contains more numbers?

12. The vertices of a convex 1991-gon are enumerated with integers from 1 to 1991. Each side and diagonal of the 1991-gon is coloured either red or blue. Prove that, for an arbitrary renumeration of vertices, one can find integers k and l such that the line connecting vertices with numbers k and l before the renumeration has the same colour as the line between the vertices having these numbers after the renumeration.

13. An equilateral triangle is divided into 25 congruent triangles enumerated with numbers from 1 to 25. Prove that one can find two triangles having a common side and with the difference of the numbers assigned to them greater than 3.

14. A castle has a number of halls and n doors. Every door leads into another hall or outside. Every hall has at least two doors. A knight enters the castle. In any hall, he can choose any door for exit except the one he just used to enter that hall. Find a strategy allowing the knight to get outside after visiting no more than $2n$ halls (a hall is counted each time it is entered).

15. In each of the squares of a chess board an arbitrary integer is written. A king starts to move on the board. As the king moves, 1 is added to the number in each square it "visits". Is it always possible to make the numbers on the chess board:

    (a) all even;

    (b) all divisible by 3;

    (c) all equal?

16. Let two circles C_1 and C_2 (with radii r_1 and r_2) touch each other externally, and let l be their common tangent. A third circle C_3 (with radius $r_3 < \min(r_1, r_2)$) is externally tangent to the two given circles and tangent to the line l. Prove that

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}.$$

17. Let the coordinate planes have the reflection property. A beam falls onto one of them. How does the final direction of the beam after reflecting from all three coordinate planes depend on its initial direction?

18. Is it possible to put two tetrahedra of volume $\frac{1}{2}$ without intersection into a sphere with radius 1?

19. Let's expand a little bit three circles, touching each other externally, so that three pairs of intersection points appear. Denote by A_1, B_1, C_1 the three so obtained "external" points and by A_2, B_2, C_2 the corresponding "internal" points. Prove the equality

$$|A_1B_2| \cdot |B_1C_2| \cdot |C_1A_2| = |A_1C_2| \cdot |C_1B_2| \cdot |B_1A_2|.$$

20. Consider two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the graph of the function $y = \frac{1}{x}$ such that $0 < x_1 < x_2$ and $|AB| = 2 \cdot |OA|$ (O is the reference point, i.e., $O(0,0)$). Let C be the midpoint of the segment AB. Prove that the angle between the x-axis and the ray OA is equal to three times the angle between x-axis and the ray OC.