Baltic Way 2007
Copenhagen, November 3, 2007

Version: English

Time allowed: 4 ½ hours.
During the first 30 minutes, questions may be asked.
Tools for writing and drawing are the only ones allowed.

1. For a positive integer n consider any partition of the set $\{1, 2, \ldots, 2n\}$ into n two-element subsets $P_1, P_2, \ldots, P_n$. In each subset P_i, let p_i be the product of the two numbers in P_i. Prove that

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} < 1.$$

2. A sequence of integers a_1, a_2, a_3, … is called *exact* if $a_n^2 - a_m^2 = a_{n-m}a_{n+m}$ for any $n > m$.

Prove that there exists an exact sequence with $a_1 = 1$, $a_2 = 0$, and determine a_{2007}.

3. Suppose that F, G, H are polynomials of degree at most $2n+1$ with real coefficients such that:

(1) For all real x we have

$$F(x) \leq G(x) \leq H(x).$$

(2) There exist distinct real numbers x_1, x_2, …, x_n such that

$$F(x_i) = H(x_i) \qquad \text{for } i = 1, 2, \ldots, n.$$

(3) There exists a real number x_0 different from x_1, x_2, …, x_n such that

$$F(x_0) + H(x_0) = 2G(x_0).$$

Prove that $F(x) + H(x) = 2G(x)$ for all real numbers x.

4. Let a_1, a_2, …, a_n be positive real numbers, and let $S = a_1 + a_2 + \cdots + a_n$. Prove that

$$(2S+n)(2S + a_1a_2 + a_2a_3 + \cdots + a_na_1) \geq 9\left(\sqrt{a_1a_2} + \sqrt{a_2a_3} + \cdots + \sqrt{a_na_1}\right)^2.$$

5. A function f is defined on the set of all real numbers except 0 and takes all real values except 1. It is also known that

$$f(xy) = f(x)f(-y) - f(x) + f(y)$$

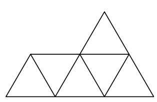
for any $x, y \neq 0$, and that

$$f(f(x)) = \frac{1}{f(\frac{1}{x})}$$

for any $x \notin \{0, 1\}$. Determine all such functions f.

6. Freddy writes down numbers $1, 2, \ldots, n$ in some order. Then he makes a list of all pairs (i, j) such that $1 \leq i < j \leq n$ and the ith number is bigger than the jth number in his permutation. After that, Freddy repeats the following action while possible: choose a pair (i, j) from the current list, interchange the ith and the jth number in the current permutation, and delete (i, j) from the list. Prove that Freddy can choose pairs in such an order that, after the process finishes, the numbers in the permutation are ordered ascendingly.

7. A *squiggle* is composed of six regular triangles with side length 1 as shown in the figure below. Determine all possible integers n such that a regular triangle with side length n can be fully covered with squiggles (rotations and reflections of squiggles are allowed, overlappings are not).



8. Call a set A of integers *non-isolated*, if for every $a \in A$ at least one of the numbers $a-1$ and $a+1$ also belongs to A. Prove that the number of five-element non-isolated subsets of $\{1, 2, \ldots, n\}$ is $(n-4)^2$.

9. A society has to elect a board of governors. Each member of the society has chosen 10 candidates for the board, but he will be happy if at least one of them will be on the board. For each six members of the society there exists a board consisting of two persons making all of these six members happy. Prove that a board consisting of 10 persons can be elected making every member of the society happy.

10. We are given an 18×18 table, all of whose cells may be black or white. Initially all the cells are coloured white. We may perform the following operation: choose one column or one row and change the colour of all cells in this column or row. Is it possible by repeating the operation to obtain a table with exactly 16 black cells?

11. In triangle ABC let AD, BE and CF be the altitudes. Let the points P, Q, R and S fulfil the following requirements:

(1) P is the circumcentre of triangle ABC.

(2) All the segments PQ, QR and RS are equal to the circumradius of triangle ABC.

(3) The oriented segment PQ has the same direction as the oriented segment AD. Similarly, QR has the same direction as BE, and RS has the same direction as CF.

Prove that S is the incentre of triangle ABC.

12. Let M be a point on the arc $\widehat{AB}$ of the circumcircle of the triangle ABC which does not contain C. Suppose that the projections of M onto the lines AB and BC lie on the sides themselves, not on their extensions. Denote these projections by X and Y, respectively. Let K and N be the midpoints of AC and XY, respectively. Prove that $\angle MNK = 90^\circ$.

13. Let $t_1, t_2, \ldots, t_k$ be different straight lines in space, where $k > 1$. Prove that points P_i on t_i, $i = 1, \ldots, k$, exist such that P_{i+1} is the projection of P_i on t_{i+1} for $i = 1, \ldots, k-1$, and P_1 is the projection of P_k on t_1.

14. In a convex quadrilateral $ABCD$ we have $\angle ADC = 90^\circ$. Let E and F be the projections of B onto the lines AD and AC, respectively. Assume that F lies between A and C, that A lies between D and E, and that the line EF passes through the midpoint of the segment BD. Prove that the quadrilateral $ABCD$ is cyclic.

15. The incircle of the triangle ABC touches the side AC at the point D. Another circle passes through D and touches the rays BC and BA, the latter at the point A. Determine the ratio AD/DC.

16. Let a and b be rational numbers such that $s = a + b = a^2 + b^2$. Prove that s can be written as a fraction where the denominator is relatively prime to 6.

17. Let x, y, z be positive integers such that $\frac{x+1}{y} + \frac{y+1}{z} + \frac{z+1}{x}$ is an integer. Let d be the greatest common divisor of x, y and z. Prove that $d \leq \sqrt[3]{xy + yz + zx}$.

18. Let a, b, c, d be non-zero integers, such that the only quadruple of integers (x, y, z, t) satisfying the equation

$$ax^2 + by^2 + cz^2 + dt^2 = 0$$

is $x = y = z = t = 0$. Does it follow that the numbers a, b, c, d have the same sign?

19. Let r and k be positive integers such that all prime divisors of r are greater than 50.

A positive integer, whose decimal representation (without leading zeroes) has at least k digits, will be called *nice* if every sequence of k consecutive digits of this decimal representation forms a number (possibly with leading zeroes) which is a multiple of r.

Prove that if there exist infinitely many nice numbers, then the number $10^k - 1$ is nice as well.

20. Let a and b be positive integers, $b < a$, such that $a^3 + b^3 + ab$ is divisible by $ab(a-b)$. Prove that ab is a perfect cube.