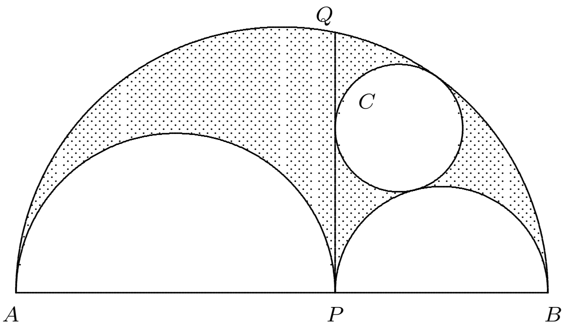
"Baltic Way – 96" Mathematical Team Contest
Valkeakoski (Finland), November 3, 1996

1. Let α be the angle between two lines containing the diagonals of a regular 1996-gon, and let $\beta \neq 0$ be another such angle. Prove that α/β is a rational number.

2. In the figure below, you see three half-circles. The circle C is tangent to two of the half-circles and to the line PQ perpendicular to the diameter AB. The area of the shaded region is 39π, and the area of the circle C is 9π. Find the length of the diameter AB.



3. Let $ABCD$ be a unit square and let P and Q be points in the plane such that Q is the circumcentre of triangle BPC and D be the circumcentre of triangle PQA. Find all possible values of the length of segment PQ.

4. $ABCD$ is a trapezium ($AD \parallel BC$). P is the point on the line AB such that $\angle CPD$ is maximal. Q is the point on the line CD such that $\angle BQA$ is maximal. Given that P lies on the segment AB, prove that $\angle CPD = \angle BQA$.

5. Let $ABCD$ be a cyclic convex quadrilateral and let r_a, r_b, r_c, r_d be the radii of the circles inscribed in the triangles BCD, ACD, ABD, ABC, respectively. Prove that $r_a + r_c = r_b + r_d$.

6. Let a, b, c, d be positive integers such that $ab = cd$. Prove that $a + b + c + d$ is not a prime.

7. A sequence of integers $a_1, a_2, \ldots$ is such that $a_1 = 1$, $a_2 = 2$ and for $n \geq 1$,

$$a_{n+2} = \begin{cases} 5a_{n+1} - 3a_n & \text{if } a_n \cdot a_{n+1} \text{ is even,} \\ a_{n+1} - a_n & \text{if } a_n \cdot a_{n+1} \text{ is odd.} \end{cases}$$

Prove that $a_n \neq 0$ for all n.

8. Consider the sequence: $x_1 = 19$, $x_2 = 95$, $x_{n+2} = \operatorname{lcm}(x_{n+1}, x_n) + x_n$, for $n > 1$, where $\operatorname{lcm}(a, b)$ means the least common multiple of a and b. Find the greatest common divisor of x_{1995} and x_{1996}.

9. Let n and k be integers, $1 \leq k < n$. Find an integer b and a set A of n integers satisfying the following conditions:
(i) No product of $k - 1$ distinct elements of A is divisible by b.
(ii) Every product of k distinct elements of A is divisible by b.
(iii) For all distinct a, a' in A, a does not divide a'.

10. Denote by $d(n)$ the number of distinct positive divisors of a positive integer n (including 1 and n). Let $a > 1$ and $n > 0$ be integers such that $a^n + 1$ is a prime. Prove that $d(a^n - 1) \geq n$.

11. Real numbers $x_1, x_2, \ldots, x_{1996}$ have the following property: For any polynomial W of degree 2 at least three of the numbers $W(x_1), W(x_2), \ldots, W(x_{1996})$ are equal. Prove that at least three of the numbers $x_1, x_2, \ldots, x_{1996}$ are equal.

12. Let S be a set of integers containing the numbers 0 and 1996. Suppose further that any integer root of any non-zero polynomial with coefficients in S also belongs to S. Prove that -2 belongs to S.

13. Consider the functions f defined on the set of integers such that

$$f(x) = f(x^2 + x + 1)\,,$$

for all integer x. Find (a) all even functions, (b) all odd functions of this kind.

14. The graph of the function $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ (where $n > 1$) intersects the line $y = b$ at the points $B_1, B_2, \ldots, B_n$ (from left to right), and the line $y = c$ ($c \neq b$) at the points $C_1, C_2, \ldots, C_n$ (from left to right). Let P be a point on the line $y = c$, to the right to the point C_n. Find the sum

$$\cot(\angle B_1 C_1 P) + \ldots + \cot(\angle B_n C_n P)\,.$$

15. For which positive real numbers a, b does the inequality

$$x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1 \geq x_1^a x_2^b x_3^a + x_2^a x_3^b x_4^a + \ldots + x_n^a x_1^b x_2^a$$

hold for all integers $n > 2$ and positive real numbers $x_1, \ldots, x_n$?

16. On an infinite checkerboard two players alternately mark one unmarked cell. One of them uses $\times$, the other $\circ$. The first who fills a 2×2 square with his symbols wins. Can the player who starts always win?

17. Using each of the eight digits 1, 3, 4, 5, 6, 7, 8 and 9 exactly once, a three-digit number A, two two-digit numbers B and C, $B < C$, and a one digit number D are formed. The numbers are such that $A + D = B + C = 143$. In how many ways can this be done?

18. The jury of an olympiad has 30 members in the beginning. Each member of the jury thinks that some of his colleagues are competent, while all the others are not, and these opinions do not change. At the beginning of every session a voting takes place, and those members who are not competent in the opinion of more than one half of the voters are excluded from the jury for the rest of the olympiad. Prove that after at most 15 sessions there will be no more exclusions. (Note that nobody votes about his own competence.)

19. Four heaps contain 38, 45, 61 and 70 matches respectively. Two players take turn choosing any two of the heaps and take some non-zero number of matches from one heap and some non-zero number of matches from the other heap. The player who cannot make a move, loses. Which one of the players has a winning strategy?

20. Is it possible to partition all positive integers into disjoint sets A and B such that

(i) no three numbers of A for an arithmetic progression,

(ii) no infinite non-constant arithmetic progression can be formed by numbers of B?